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ANALYSIS OF DEHN ALGORITHM BY CRITICAL PAIRS

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RESUME

Le formalisme des paires critiques et des ensembles symétrisés de règles de récriture fournit de bons outils pour l'analyse de l'algorithmes de Dehn. Une telle analyse donne une caractérisation des groupes de "petites réduction" au moyen des diagrammes de Cayley. Les conditions classiques C(p) et T(q) sont obtenues comme cas particuliers.

Analysis of Dehn algorithm by critical pairs

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ABSTRACT

Symmetrized sets of replacement rules and critical pairs in reductions provide a good framework for the analysis of Dehn algorithm for the word problem. This analysis gives a geometrical characterization of small cancellation groups by means of Cayley diagrams, including as special cases the usual **C(*p*)** and **T(*q*)** conditions.

1. Introduction

This paper analyses Dehn algorithm for the word problem in groups by rewriting techniques. Since Dehn's first investigations [Deh11, Deh12], the class of small cancellation groups has been intensively studied. Two styles of proofs for the fundamental result exist: Greendlinger's one [Gree60a, b] based on word combinatorics, and Lyndon's one [Lyn66] by more geometrical means. The proof presented here stands half-way between these two trends. It begins with combinatorics on relations that naturally give geometrical interpretations. The basic tool is rewrite rules, a useful and unifying concept emerged from symbolic computation [LeC85]. Bücken [Bü79a, b] has first given a proof based on this technique. With respect to his work, the present paper emphases on rules configurations reducing critical pairs (minimal divergent points in derivations) and having simple geometric meaning in terms of Cayley diagrams. The number of such diagrams is infinite but finitely generated by the defining relations of the group. The word problem algorithms described by rules refines Dehn algorithm in that length-preserving reductions are allowed. For example the group *G=(A,B,C;ABC=CBA)* does not satisfy the classical conditions and non-confluent Dehn reductions exist [Gre60a]. However, it falls under our conditions. Roughly speaking, these conditions are the classical C(4) one together with non-existence of diagrams like the following one, where each region corresponds to a defining relation:

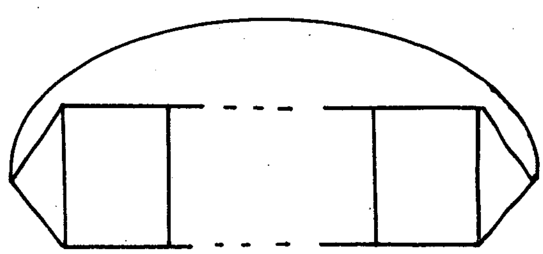

Fig. 1

The next section is devoted to a detailed analysis of the *symmetrization* algorithm, a restricted case of a well-known completion procedure [Knu70] whose application to groups can be found in [LeC85]. The study of rules configurations involved in reductions of *critical* words is the matter of the third section. The proof of the main theorem and a discussion on some examples concludes the paper.

2. The Symmetrization

A presentation of a group **G** is a pair (G,S), G a set of generators, S a set of defining relations from the free monoid $(G\cup G^{-1})^*$. These two sets are supposed to be finite. Knowledge about the following notions is assumed: freely reduced words, cyclically reduced words, symmetrized sets of relations (ssr), Cayley diagram of a presentation and S-diagrams, S a ssr, pieces between relations and conditions $\mathbf{C}'(1/p)$,$\mathbf{C}(p)$ and $\mathbf{T}(q)$, p,q positive integers. For definitions we refer to chapter V of [Lyn77]. The usual interpretation of conditions $\mathbf{C}(p)$ and $\mathbf{T}(q)$ is localized to a single region and vertex respectively. We shall define diagrams concerned with sequences of regions.

Let us define the basic notions of ***rewriting*** [LeC85]. A ***rule*** is a pair (λ,ρ) of freely reduced words from $G\cup G^{-1}$, noted $\lambda\rightarrow\rho$. A rewriting system R is a finite set of rules. The word w R-reduces on w', $w\underset{R}{\rightarrow}w'$, iff there exists a rule $\lambda\rightarrow\rho$ in R and words a,b such that $w=a\lambda b$ and $w'=a\rho b$. The reflexive-transitive closure of this relation is noted $\overset{*}{\underset{R}{\rightarrow}}$. For example the set $F_G=\{aa^{-1}\rightarrow 1, a^{-1}a\rightarrow 1 \mid a\in G\}$ defines the usual free cancellation (the index G will be dropped when clear from the context). Successive reductions will be represented by concatenation of arrows. The reduction relations will be supposed well-founded. This is achieved if all rules $\lambda\rightarrow\rho$ satisfy an inequality $\lambda>\rho$ where $>$ is a partial (non-reflexive) ordering such that:

i) $\forall w,w' \;\; w>w' \Rightarrow \forall u,v \;\; uwv>uw'v$.

ii) $\forall u,v \;\; |u|>|v| \Rightarrow u>v$.

iii) The partial ordering $>$ is well-founded.

Throughout the paper, all orderings will satisfy these three requirements. Let $\lambda A\rightarrow\rho$ and $A\mu\rightarrow\nu$ be two rules, $A\neq 1$, the pair $(\rho\mu,\lambda\nu)$ is called a ***critical pair*** between these rules. It is resolved or confluent iff there exists a word ω such that $\rho\mu\overset{*}{\underset{R}{\rightarrow}}\omega$ and $\lambda\nu\overset{*}{\underset{R}{\rightarrow}}\omega$.

Of particular importance in this paper are the critical pairs between a rule in F and another one in R. If $a_1\cdots a_n\rightarrow b_1\cdots b_m$ is a rule, $a_i,b_j\in G\cup G^{-1}$, we define its ***normal pairs*** to be the three pairs
$(a_1\cdots a_{n-1}\,,\, b_1\cdots b_m a_n^{-1})$, $(a_2\cdots a_n\,,\, a_1^{-1}b_1\cdots b_m)$,
and $(a_n^{-1}\cdots a_1^{-1}\,,\, b_m^{-1}\cdots b_1^{-1})$. They are the critical pairs between the rule and the canonical system of groups (cf. [LeC85]). The symmetrization algorithm resolves some of the reduction ambiguities by transforming normal pairs in new rules according to a given ordering. We assume that the ordering is total, $R(w)$ denotes a normal form of w under the rule set $R\cup F_G$.

Group Symmetrization Algorithm

Input E : A finite set of defining equations. $<$: An ordering.

$R_0:=\phi$; The set of rules.

$E_0:=E$; The set of waiting equations.

$RR_0:=\phi$;The set of deleted rules.

$S_0:=\phi$; The set of rules whose normal pairs are not computed.
$i:=p:=0$;The step counter and the rules counter.

Loop { **If** $RR_i \neq \phi$ **Then** { *choose a pair* $(u_0,v_0)\in RR_i$; $RR_{i+1}:=RR_i - \{(u_0,v_0)\}$ }
If $E_i \neq \phi$ **Then** { *choose a pair* $(u_0,v_0)\in E_i$; $E_{i+1}:=E_i - \{(u_0,v_0)\}$ };
If $RR_i \cup E_i \neq \phi$ **Then** *Create_Rule*$(R_i(u_0),R_i(v_0))$
Else { **If** $S_i=\phi$ **Then** *Success*
Else { *choose a rule k in* S_i; $S_{i+1}:=S_i-\{k\}$;
Let E_{i+1} *be the set of normal pairs of rule* k; $i:=i+1$ }}}

where Create_Rule(u,v) =
If $u=v$ **Then** $i:=i+1$
Else { **If** $u>v$ **Then** { $\lambda:=u,\ \rho:=v$ }
Else $u<v$ **Then** { $\lambda:=v,\ \rho:=u$ };
$p:=p+1$;
$K:=\{k \mid k:\alpha\rightarrow\beta\in R_i$ *and* α *is reducible by* $p:\lambda\rightarrow\rho\}$;
$RR_{i+1}:=\{(\alpha,\beta) \mid k:\alpha\rightarrow\beta\in R_i,\ k\in K\} \cup RR_i$;
$R_{i+1}:=\{k:\alpha\rightarrow\beta' \mid k:\alpha\rightarrow\beta\in R_i,\ k\notin K,\ \beta'=(R_i\cup\{p:\lambda\rightarrow\rho\})(\beta)\}$
$\cup\{p:\lambda\rightarrow\rho\}$;
$S_{i+1}:=S_i \cup\{p\}$; $i:=i+1$ } ∎

The first result about this algorithm is technical, but important in the sequel. It expresses a continuity between ***stable*** rules (which are never reduced neither eliminated).

Lemma 1
Let $k: a\lambda b\rightarrow\rho$ *be a stable rule,* $a,b\in G\cup G^{-1}$. *If the presentation* G *satisfies* C'(2), *then*

i) $l: a^{-1}\rho\rightarrow\lambda b$ *is stable or* $a^{-1}\rho \underset{n}{\rightarrow} . \overset{*}{\underset{F}{\rightarrow}} \lambda b$, n *a stable rule*

ii) $m: \rho b^{-1}\rightarrow a\lambda$ *is stable or* $\rho b^{-1} \underset{n}{\rightarrow} . \overset{*}{\underset{F}{\rightarrow}} a\lambda$, n *a stable rule.*

Proof. By symmetry, we only prove i). The computation of normal pairs generates at some iteration the pair $(a^{-1}\rho,\lambda b)$. As k is a stable rule, the rule $i: \lambda b \rightarrow a^{-1}\rho$ cannot be generated. This yields only two cases, either the first expressed in i), or the pair has been resolved. In this latter case, as λb and ρ are irreducible by the stability of k, there exists $\alpha\neq 1$ and a stable rule $n: a^{-1}\mu\rightarrow\nu$ with $a^{-1}\rho=a^{-1}\mu\alpha$. The stability of n follows from the fact that every left member from R_i is R_j-reducible for all $j\geq i$. But our rules are not length-increasing, thus $|a^{-1}\mu|\geq|\nu|$. That is, by assumption **C**'(2), two of the relations deduced from rules n and k are equal: $a^{-1}\mu\nu^{-1}=a^{-1}\rho b^{-1}\lambda^{-1}$. We have the reductions

$$a^{-1}\rho=a^{-1}\mu\alpha \underset{n}{\rightarrow} \lambda b\, a^{-1}\alpha \overset{*}{\underset{F}{\rightarrow}} \lambda b$$

which proves i) ∎

Proposition 2
Given both a group presentation G *and an ordering* >, *the symmetrization algorithm always terminates. Let* Γ *be the set of rules computed, then if* $k: \lambda\rightarrow\rho\in\Gamma$, *we have :*
- *The words* $\lambda\rho^{-1}$, $\lambda^{-1}\rho$, $\rho\lambda^{-1}$ *and* $\rho^{-1}\lambda$ *are relations.*
- *If* $|\lambda|+|\rho|=2p+1$ *then* $|\lambda|=p+1, |\rho|=p$.

- *If* $|\lambda|+|\rho|=2p$ *then* $|\lambda|=p=|\rho|$ *or* $|\lambda|=p+1, |\rho|=p-1$.

If G *satisfies* C'(2) *then*

- *The set* $S=\{\lambda\rho^{-1}, \rho\lambda^{-1} | k: \lambda \longrightarrow \rho\in\Gamma\}$ *is the ssr of defining relations.*
- *If* $UV\in S$ *and* $|U|>|V|$ *then* $U \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} V^{-1}$.
- ***A non-confluent critical pair results from a superposition on a piece between two relations.***

Proof. Observe first that reductions always halt as the given ordering is well-founded. We first prove the termination of the procedure. For each rule $k: \lambda \longrightarrow \rho\in R_i$, and all $j\geq i$, there exists another rule $l: \alpha \longrightarrow \beta$ in R_j that reduces λ (possibly $l=k$). Next, the number of words that may appear in any rule is finite: as only normal pairs are computed and rules are inter-reduced, these words are subwords of the symmetrized set of defining relations. Therefore only a finite number of rules can be generated. At some iteration, no new rule is created. Afterwards, the pair of numbers of marked rules and waiting equations lexicographically decreases at each iteration. Therefore, the algorithm stops.

Let Γ be the set of rules computed. If $k: \lambda \longrightarrow \rho\in\Gamma$, then both left and right hand sides are in F-canonical form. If the word $\lambda\rho^{-1}$ was not in F-normal form, then rule k would be of the type $\lambda' a \longrightarrow \rho' a$, $a\in G\cup G^{-1}$. One of its normal pair is $(\lambda', \rho' a a^{-1})$, whith F-normal form (λ', ρ'). But all three cases $\lambda' \longrightarrow \rho'$, $\rho' \longrightarrow \lambda'$ in Γ, or $\lambda'=\rho'$ contradict either the Γ-irreducibility of rule k or the fact that $\lambda\neq\rho$. For the same reason, the word $\lambda\rho^{-1}$ is cyclically reduced. And the remaining cases are similar.

Set $\lambda=a\lambda'$ in rule k, then the normal pair $(\lambda', a^{-1}\rho)$ is confluent as long as the symmetrization halts. Therefore, λ' being Γ-irreducible, we must have $|a^{-1}\rho|\geq|\lambda'|$. But $|a\lambda'|\geq|\rho|$ by length hypothesis, so that $0\leq|a\lambda'|-|\rho|\leq 2$. Thus, if $|\lambda|+|\rho|=2p$, we have $p\leq|\lambda|\leq p+1$, and if $|\lambda|+|\rho|=2p+1$, we have $p+\frac{1}{2}\leq|\lambda|\leq p+\frac{3}{2}$. This proves the length assertions.

First, let aR be a word in S, $a\in G\cup G^{-1}$. We know from the first assertion that aR is a relation. By definition of S, we get $R^{-1}a^{-1}\in S$. Thus we only need to prove $Ra\in S$. Let first aR be of the form $\lambda\rho^{-1}$, then $\lambda=a\lambda'$ and lemma 1 implies two cases, either $l_1: a^{-1}\rho \longrightarrow \lambda'\in\Gamma$ or there exists $\alpha\neq 1$ such that $\rho=\rho'\alpha$ and l_2: $a^{-1}\rho' \longrightarrow \lambda'\alpha^{-1}\in\Gamma$. Involving the definition of the set S on these two rules gives $Ra\in S$. Second, we have $aR=\rho\lambda^{-1}$. We apply now two times lemma 1. First there exists a rule l_1:$\lambda' b \longrightarrow a\rho'\in\Gamma$ where $\lambda' b=\lambda$, $b\in G\cup G^{-1}$. It follows that l_2:$a\rho' b^{-1} \longrightarrow \lambda'\in\Gamma$. Applying once more the lemma we get the rule we were seeking for, l_3:$a^{-1}\lambda' \longrightarrow \rho' b^{-1}$. By definition of S, we get in this latter case $Ra\in S$. Thus this set is precisely the symmetrized set of defining relations. Note that we omit in the proof the second cases of lemma 1. This case is exactly the one we studied modulo subword's labels. Now, if $UV\in S$ and $|U|>|V|$, then

— $UV=\lambda\rho^{-1}$, the length identities imply $|U|\geq|\lambda|$, thus there exists U' such that $\lambda U'=U$ and $U'V=\rho^{-1}$. We have the following derivations:

$$U=\lambda U' \underset{\Gamma}{\Longrightarrow} \rho U' = V^{-1}U'^{-1}U' \overset{*}{\underset{F}{\Longrightarrow}} V^{-1}$$

— $UV=\rho\lambda^{-1}$ implies $|U|>|\rho|$ and $\exists U'$, $\exists a\in G\cup G^{-1}$ with $\rho a U'=U$ and $aU'V=\lambda^{-1}$. Then, by lemma 1, we have the following derivations:

$$U=\rho a U' \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} V^{-1}U'^{-1}U' \overset{*}{\underset{F}{\Longrightarrow}} V^{-1}$$

Now, to conclude the proof, let (AE, DC) be a critical pair:

$$\begin{cases} k: AB \longrightarrow D \\ l: BC \longrightarrow E \end{cases} \quad B \neq 1$$

such that the common non-empty subword B is not a piece from S. Necessarily, two cyclic permutations of the relations associated to rules k and l are equal. For example:

$$\begin{aligned} & BD^{-1}A &&= BCE^{-1} \\ \Longrightarrow\ & D^{-1}A &&= CE^{-1} \\ \Longrightarrow\ & DD^{-1}AE &&= DCE^{-1}E \\ \Longrightarrow\ & AE &&=_F DC \end{aligned}$$

The last equality implies that the critical pair is resolved ∎

Thus, to a rule $\lambda \longrightarrow \rho$ we can associate a set of relations, namely $\lambda\rho^{-1}$, its inverse and all cyclic permutations. Prop.2 claims that unresolved critical pairs are computed by superposition on a piece. If the ssr S satisfies $\mathbf{C}'(4)$, they are Γ-irreducible.

Lemma 3

If S satisies $\mathbf{C}'(4)$, then every critical pair computed from a piece is, perhaps after F-reduction, in F,Γ-normal form.

Proof Pick up two rules that superpose on a piece. The match includes at most the whole piece, which splits in three words: the matching subword B, non-empty, a prefix A and a suffix C, both possibly empty:

$$\mathrm{I}\begin{cases} k: & \alpha AB \longrightarrow \beta C^{-1} \\ l: & BC\gamma \longrightarrow A^{-1}\delta \end{cases}$$

The critical pair is $P=(\beta C^{-1}C\gamma, \alpha AA^{-1}\delta)$. It reduces by F in $Q=(\beta\gamma, \alpha\delta)$, words $\beta\gamma$ and $\alpha\delta$ being F-irreducible. Suppose the word $\beta\gamma$ Γ-reducible, the other case being similar. Both β and γ are F,Γ-irreducible, also there exists a Γ-rule m: $\lambda\rho \longrightarrow \tau$, such that λ is a proper suffix of β and ρ a proper prefix of γ. We have three cases:

1) λ is not a piece between rules m and k. Two associated relations are identical:

$$\lambda\rho\tau^{-1} = \lambda C^{-1}B^{-1}A^{-1}\alpha^{-1}\beta_1 \quad \text{where} \quad \beta_1\lambda = \beta \tag{1}$$

Then C is the empty word. Otherwise, ρ and C^{-1} sharing a common prefix by (1), rule l would not be F-reduced at subword $C\gamma$. The same remark implies that B or λ is the empty word, this contradicts our assumptions.

2) ρ is not a piece between m and l, the same deduction remains valid:

$$\rho\tau^{-1}\lambda = \rho\gamma_1\delta^{-1}ABC \quad \text{where} \quad \rho\gamma_1 = \gamma$$

Actually, the words λ and C share a common suffix, the F,Γ-irreducibility of βC^{-1} (as right member) implies $C=1$. But k would then be of the type $Ua \longrightarrow Va$, this is impossible by Prop.2.

3) λ and ρ must be pieces. By $\mathbf{C}'(4)$ assumption $|\lambda\rho| < \frac{1}{2}|\lambda\rho\tau^{-1}|$. But rules are not length-increasing, contradiction. This achieves the proof ∎

We keep in mind that, together with the $\mathbf{C}'(2)$ condition, what we really do not want is the existence in Γ of the two following configurations:

$$\mathrm{L}\begin{cases} \alpha AB \longrightarrow \beta\lambda C^{-1} \\ BC\rho\gamma \longrightarrow A^{-1}\delta \\ \lambda\rho \longrightarrow \tau \end{cases} \qquad \mathrm{R}\begin{cases} \alpha\lambda AB \longrightarrow \beta C^{-1} \\ BC\gamma \longrightarrow A^{-1}\rho\delta \\ \lambda\rho \longrightarrow \tau \end{cases} \qquad \text{Pieces } ABC, \lambda, \rho.$$

The symmetrization algorithm generates a normalized or symmetrized presentation Γ with nice syntactic properties. Note that the notion of symmetrized set of rules or ***normalized presentation*** contains the two notions of Dehn algorithm and of symmetrized set of relations. The points of divergence in reductions are critical pairs computed from pieces. The next section is devoted to the analysis of these points. So far they are irreducible. Thus we shall force their reduction by the concatenation of ***contexts***. But let us see first two examples of Γ rewriting systems, the latter is **C'**(2), the former is not.

The defining relation is $TTA=TTB=1$. The piece TT in fact defines the new relation BA^{-1}. Moreover, the relation TTA is no more of the form $\lambda\rho^{-1}$ or $\rho\lambda^{-1}$ (cf. Prop.2), we have lost the fair properties of Γ:

$$\left\{\begin{array}{ccccccc} A & \rightarrow & B & & BT & \rightarrow & T^{-1} \\ A^{-1} & \rightarrow & B^{-1} & & TB & \rightarrow & T^{-1} \\ TT & \rightarrow & B^{-1} & & B^{-1}T^{-1} & \rightarrow & T \\ T^{-1}T^{-1} & \rightarrow & B & & T^{-1}B^{-1} & \rightarrow & T \end{array}\right.$$

The following Γ set shows the necessity in the definition of S for both $\lambda\rho^{-1}$ and $\rho\lambda^{-1}$, its defining relation is $ABCD=DCBA$.

$$\left\{\begin{array}{ccccccc} DCBA & \rightarrow & ABCD & & D^{-1}C^{-1}B^{-1}A^{-1} & \rightarrow & A^{-1}B^{-1}C^{-1}D^{-1} \\ BCDA^{-1} & \rightarrow & A^{-1}DCB & & B^{-1}C^{-1}D^{-1}A & \rightarrow & AD^{-1}C^{-1}B^{-1} \\ B^{-1}A^{-1}DC & \rightarrow & CDA^{-1}B^{-1} & & BAD^{-1}C^{-1} & \rightarrow & C^{-1}D^{-1}AB \\ DA^{-1}B^{-1}C^{-1} & \rightarrow & C^{-1}B^{-1}A^{-1}D & & D^{-1}ABC & \rightarrow & CBAD^{-1} \end{array}\right.$$

The relation $ABCDA^{-1}B^{-1}C^{-1}D^{-1}$ is not $\lambda\rho^{-1}$ but $\rho\lambda^{-1}$ by the first rule, and $DCBAD^{-1}C^{-1}B^{-1}A^{-1}$ is $\lambda\rho^{-1}$ yet by the first rule, but not $\rho\lambda^{-1}$ (for both Γ sets, well-foundedness is trivial).

3. Critical Pair Reduction by Contexts

Lemma 2 claims the Γ-irreducibility of critical pairs. This section analyses their reductions forced by the concatenation of contexts. With the notations of I, a piece ABC gives a critical pair $P=(\alpha\delta,\beta\gamma)$. The members of P are F,Γ-irreducible. We study reductions of this pair under left or right contexts, seeking confluence conditions or irreducibility criterions for the word reduced by the context.

A ***left*** (resp. ***right***) context is a F,Γ-irreducible word μM (resp. $T\tau$), where M (resp. T), possibly empty, is absorbed by free cancellation, while μ (resp. τ) is a ***proper*** prefix (resp. suffix) of a Γ-rule left member. Contexts create a Γ-redex after F-cancellation of M or T. The study being symmetrical with respect to $\alpha\delta$ and $\beta\gamma$, we restrict our attention to the word $\beta\gamma$. Reductions start from $\mu M\beta\gamma$ and $\beta\gamma T\tau$.

Lemma 4

If in the reduction by a left (resp. right) context of $\beta\gamma$, β (resp. γ) is absorbed by F-reduction, then, in this context, the critical pair P is confluent.

Proof. As β and γ appear in the contexts, we rewrite them in $\mu M\beta^{-1}$ and $\gamma^{-1}T\tau$. For left contexts, we have

$$\mu M\beta^{-1}\beta\gamma \xrightarrow[F]{*} \mu M\gamma \tag{2}$$

Therefore, the word $\mu M\beta^{-1}\alpha\delta$ is also reducible by a rule in Γ. We have, by rule k, $\beta^{-1}\alpha=_G C^{-1}B^{-1}A^{-1}$, and the second member of this equality being a piece, the hypothesis **C'**(2) and Prop.2 imply $\beta^{-1}\alpha \xrightarrow[F,\Gamma]{*} C^{-1}B^{-1}A^{-1}$. Writing $BC=aU$, $a\in G\cup G^{-1}$ $(B\neq 1)$, lemma 1 applied to rule l gives $a^{-1}A^{-1}\delta \xrightarrow[F,\Gamma]{*} U\gamma$.

Consequently, we have the derivations:

$$\mu M\beta^{-1}\alpha\delta \xrightarrow[F,\Gamma]{*} \mu MC^{-1}B^{-1}A^{-1}\delta = \mu MU^{-1}a^{-1}A^{-1}\delta \xrightarrow[F,\Gamma]{*} \mu M\gamma.$$

Together with (2), this establishes the confluence.
The other case is similar, let us just give the reductions:

$\beta\gamma\gamma^{-1}T\tau \xrightarrow{*} \beta T\tau$

and $\alpha\delta\gamma^{-1}T\tau \xrightarrow[F,\Gamma]{*} \alpha ABCT\tau$ by Prop.2.

$\longrightarrow \beta C^{-1}CT\tau$ by rule k.

$\xrightarrow[F]{*} \beta T\tau$ ∎

From now on, we suppose that the two words M and T do not completely absorb their neighbour β or γ. We first detail the action of left contexts. We may write $\beta=M^{-1}\beta_1$, so that $\mu M\beta\gamma \xrightarrow[F]{*} \mu\beta_1\gamma$. The next reduction is performed by the Γ-rule $m: \mu\mu_1 \longrightarrow \nu\nu_1$. As usual the word ν_1 points out a possible F-reduction. The study splits in two cases, according to the fact that M is empty or not. Let us analyze the latter case. For this purpose, we assume the

Hypothesis 1 : The presentation **G** satisfies **C**'(4).

Consequently, if $PQ\zeta$ is a relation where P,Q are pieces, $\zeta^{-1} \xrightarrow[F,\Gamma]{*} PQ$.

Lemma 5

Given a reduction of $\beta\gamma$ by a left context with F-reduction, then

- *When β is cancelled by $F\Gamma$-reductions, the critical pair is confluent in this context.*
- *Otherwise $\beta\gamma$ reduces to a word φ, F-irreducible, Γ-reducible only with the following configuration:*

$$\textbf{LL}\left\{\begin{array}{lll} k: \alpha AB & \longrightarrow & M^{-1}\mu_1\nu_1^{-1}\beta_2C^{-1} \\ l: BC\rho\gamma_1 & \longrightarrow & A^{-1}\delta \\ m: \mu\mu_1 & \longrightarrow & \nu_2\lambda\nu_1 \\ n: \lambda\beta_2\rho & \longrightarrow & \omega \\ \textit{Pieces}: & & ABC,\ \mu_1\nu_1^{-1},\ \beta_2,\ \lambda,\ \rho. \end{array}\right.$$

Proof.
Case 1, the reductions do not cancel a proper subword of β. We may write $\beta_1=\mu_1\nu_1^{-1}\beta_2$ where $\beta_2\neq 1$. Then

$$\mu M\beta\gamma \xrightarrow[F]{*} \mu\mu_1\nu_1^{-1}\beta_2\gamma \xrightarrow[m]{} \nu\nu_1\nu_1^{-1}\beta_2\gamma \xrightarrow[F]{*} \nu\beta_2\gamma=\varphi.$$

The word φ is in F-normal form. First, the word μ_1 is a piece between rules k and m. Otherwise we could write:

$$\mu_1\nu_1^{-1}\nu\mu=\mu_1\nu_1^{-1}\beta_2C^{-1}B^{-1}A^{-1}\alpha^{-1}M^{-1}$$

This identity implies, as $M\neq 1$, that the word μM is F-reducible, which contradicts the context F,Γ-irreducibility. Thus the piece between k and m is $\mu_1\nu_1^{-1}$.

From lemma 3, we know that both $\beta_2\gamma$ and $\nu\beta_2$ are F,Γ-irreducible (strictly speaking, in the latter case, this lemma does not apply, but its proof does as the reader may check, one occurrence of β_2 in k's right member rather than in a left one being here immaterial). Thus, a Γ-rule reducing φ

must be of the type n: $\lambda\beta_2\rho \longrightarrow \omega$ where $\lambda \neq 1$ and $\rho \neq 1$, λ suffix of ν, ρ prefix of γ. Thus we get the configuration **LL** whose known pieces are ABC and $\mu_1\nu_1^{-1}$. By three identifications of relations, we get:

—λ is a piece between rules n and m. Otherwise we have $\nu_1=1$ (rule k is irreducible and $\beta_2 \neq 1$ by hypothesis of case 1). For the same reason $\mu_1=1$, which contradicts the irreducibility of the left context μM.

—β_2 is a piece between n and k. Otherwise $C=1$ (irreducibility of l and $\rho \neq 1$). Furthermore $\nu_1=1$ (irr. of m and $\lambda \neq 1$). Then the two words λ and μ_1 share a common suffix. This is impossible by Prop.2 applied to rule m.

—ρ is a piece between n and l. Otherwise $C=1$ (irr. of k). The words B and β_2 share a common suffix, as $B \neq 1$ and $\beta_2 \neq 1$ (hypothesis of case 1), we have a contradiction (Prop.2 and rule k).

Case 2. In the reductions, the word β is entirely cancelled. If β_1 is not a piece, we may identity rules m and k:

$$\beta_1\mu_1\nu_1^{-1}\nu^{-1}\mu=\beta_1C^{-1}B^{-1}A^{-1}\alpha^{-1}M^{-1}$$

Hence, as $M \neq 1$, the context μM should be F-reducible. Thus β_1 is a piece, and the assumption **C**'(4) and Prop.2 applied to rule k show that $M\alpha \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} \beta_1C^{-1}B^{-1}A^{-1}$. The critical pair is confluent by lemma 1 applied to rule l:

$$\mu M\beta\gamma \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} \mu\beta_1\gamma$$

$$\mu M\alpha\delta \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} \mu\beta_1C^{-1}B^{-1}A^{-1}\delta \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} \mu\beta_1\gamma \quad \blacksquare$$

If $M=1$, the word $\mu\beta\gamma$ is reduced by the rule m: $\mu\mu_1 \longrightarrow \nu\nu_1$. Also the words β and μ_1 have a common prefix.

Lemma 6

Given a reduction of $\beta\gamma$ by left context without F-reduction, then

- *If the reduction is done by a piece between β and the context, we get a word φ, F-irreducible, Γ-reducible only under the configuration* **LL** *(where $M=1$).*
- *Otherwise, the critical pair is confluent in this context.*

Proof. Again we consider two subcases: whether or not the common prefix between β and μ_1 is a piece.

Case 1. The prefix is a piece. Necessarily, $\mu_1\nu_1^{-1}$ is a proper prefix of β. Otherwise, hyp. **C**'(4) and Prop.2 imply that: $\alpha \overset{*}{\underset{F,\Gamma}{\Longrightarrow}} \beta C^{-1}B^{-1}A^{-1}$ as β is now a piece. But α is irreducible as proper left member subword. We may write $\beta=\mu_1\nu_1^{-1}\beta_1$ where $\beta_1 \neq 1$. Then $\mu\beta\gamma \longrightarrow \nu\beta_1\gamma$, and we are in the first case of lemma 5 with the configuration **LL** where β_1 replaces β_2 and the word M has disappeared.

Case 2. The prefix is not a piece. We have the rule identity: $\beta C^{-1}B^{-1}A^{-1}\alpha^{-1}=\mu_1\nu_1^{-1}\nu^{-1}\mu$. If β is a proper prefix of μ_1, then there exists a word ε and $a \in G \cup G^{-1}$ such that $\mu_1=\beta a\varepsilon$. The rule identity implies the existence of C' with $C^{-1}=aC'^{-1}$. As the word $\mu\beta\gamma$ contains the left member of m, these two equalities imply that the rule l is not reduced in its left member $BC\gamma=BC'a^{-1}a\gamma_1$ where $a\gamma_1=\gamma$.

Hence μ_1 is a prefix of β: $\beta=\mu_1\beta_1$ and the rule identity becomes:

$$\mu^{-1}\nu\nu_1\mu_1^{-1}=\alpha ABC\beta_1^{-1}\mu_1^{-1} \tag{3}$$

Case 2.1. If β_1 is the empty word. Then (3) implies

$$\mu^{-1}\nu\nu_1=\alpha ABC \tag{4}$$

And we have $\nu_1=1$. Effectively, as ν_1 freely cancels a part of γ, rule l would not be F-reduced if both C and ν_1 were non-null. If $C=1$, as $B\neq 1$, rule l inter-reduced implies that $\nu_1=1$. And if $C\neq 1$, then $\nu_1=1$ necessarily. The rule configuration is now:

$$\begin{cases} l: & \alpha A b \longrightarrow \beta C^{-1} \\ k: & BC\gamma \longrightarrow A^{-1}\delta \\ m: & \mu\beta \longrightarrow \nu \end{cases}$$

In any case we have confluence of the critical pair in the context μ.

$$\mu\beta\gamma \longrightarrow \nu\gamma \tag{5}$$

If $|\alpha|>|\mu|$, then $\alpha=\mu^{-1}\alpha'$, $\alpha'\neq 1$, then $\nu=\alpha'ABC$ from (4). Both $\mu\beta\gamma$ and $\mu\alpha\delta$ reduce on $\alpha'\delta$. If $|\alpha|\leq|\mu|$, then $\mu=\mu'\alpha^{-1}$, μ' possibly empty. Then (4) gives $\mu'^{-1}\nu=ABC$ and

$$\mu\alpha\delta=\mu'\alpha^{-1}\alpha\delta \overset{*}{\underset{F}{\Longrightarrow}} \mu'\delta \tag{6}$$

If $|\mu'|\leq|A|$, then $A=\mu'^{-1}A'$, A' possibly empty. We have $\nu=A'BC$ and $\nu\gamma$ reduces to $\mu'\delta$, with (6) this gives the desired confluence. Otherwise $|\mu'|>|A|$ implies $\mu'^{-1}=A\mu''^{-1}$, $\mu''\neq 1$. The identity (4) becomes $\mu''^{-1}\nu=BC$. Then $\mu'\delta$ reduces on $\nu\gamma$ by lemma 1 applied to rule l. This with (5) proves the confluence.

Case 2.2. If β_1 is non-empty, then by (3), β_1^{-1} contains the suffix ν_1. Otherwise, there exist two words $\nu'_1\neq 1$ and γ_1 such that $\nu_1^{-1}=\beta_1\nu'^{-1}_1$ and $\gamma=\nu'^{-1}_1\gamma_1$. Once more the left member of l would not be reduced at subword $\nu'_1\nu'^{-1}_1$ on the join of C and γ.

Thus, there exists β_2 such that $\beta=\mu_1\beta_1=\mu_1\nu_1^{-1}\beta_2$, β_2 possibly empty. Then the identity (3) becomes:

$$\mu^{-1}\nu=\alpha ABC\beta_2^{-1} \tag{7}$$

Now, $\mu\beta\gamma \overset{*}{\underset{F,T}{\Longrightarrow}} \nu\beta_2\gamma$. But this last word is F-irreducible. Thus, by (7), ν or β_2 is empty. If ν is empty, then

$\mu\alpha\delta=\beta_2C^{-1}B^{-1}A^{-1}\alpha^{-1}\alpha\delta$ by (7),

$\overset{*}{\underset{F,T}{\Longrightarrow}} \beta_2\gamma$ by Prop.2.

and the critical pair is confluent in the context.

Otherwise, $\beta_2=1$. Then by (7) either ν contains the suffix BC so that $\mu\beta\gamma \overset{*}{\underset{F,T}{\Longrightarrow}} \nu\gamma$ is further reduced, or it does not and $\nu\gamma$ is irreducible. The reader may check that in both cases, there is confluence. We just sketch the proof:

$|\alpha|>|\mu| \Longrightarrow \alpha=\mu^{-1}\alpha'$, confluence on $\alpha'\delta$.

$|\alpha|\leq|\mu| \Longrightarrow \mu=\mu'\alpha^{-1}$, μ' possibly empty.

If $|A|\geq|\mu'|$, then confluence on $\mu'\delta$.

If $|A|<|\mu'|$, then confluence on $\nu\gamma$ ▪

We now study right context reductions. The main difference with the left case is that reductions of $\beta\gamma$ first reduce γ, subword of ***right*** member, while β is subword of ***left*** member. However, the two analysis are still very closed. The reduction starts with $\beta\gamma T\tau \overset{*}{\underset{F}{\Longrightarrow}} \beta\gamma_1\tau$ where $\gamma=\gamma_1T^{-1}$.

Lemma 7

Given a reduction of $\beta\gamma$ by a right context with F-reduction, then

- *If γ is cancelled by $F\Gamma$-reduction, the critical pair is confluent in this context.*
- *Otherwise $\beta\gamma$ reduces to a word ψ, F-irreducible, Γ-reducible only with a configuration:*

$$\mathbf{LR}\begin{cases} k : \alpha AB & \rightarrow \beta_1\lambda C^{-1} \\ l : BC\gamma_2\sigma_1^{-1}\tau_1 T^{-1} & \rightarrow A^{-1}\delta \\ m : \tau_1\tau & \rightarrow \sigma_1\rho\sigma_2 \\ n : \lambda\gamma_2\rho & \rightarrow \omega \\ \textbf{pieces}: \gamma_2,\ ABC, & \sigma_1^{-1}\tau_1,\ \lambda,\ \rho. \end{cases}$$

Proof. As usual we split the proof in two cases.
Case 1. The Γ-reduction by τ does not entirely absorb the word γ. Then $\gamma_1=\gamma_2\sigma_1^{-1}\tau_1$ and $m: \tau_1\tau \rightarrow \sigma_1\sigma$. The word τ_1 is a piece. Otherwise the equality $\sigma_1^{-1}\tau_1\tau\sigma^{-1}=\sigma_1^{-1}\tau_1 T^{-1}\delta^{-1}ABC\gamma_2$ implies that the context $T\tau$ would be reducible.
We have therefore the configuration **LR**. Let us give succinctly the reasons why the three words μ, ν_2, and ν are pieces:

—μ not piece $\Rightarrow C=1$. But, the words μ and B sharing a common suffix, rule k would not be reduced.

—γ_2 not piece $\Rightarrow \sigma_1=1$. But the two words τ_1 and ν share a common prefix, rule m would not be reduced.

—ν not piece $\Rightarrow \sigma_1=1$, and τ_1^{-1}, γ_2 have a common suffix, rule l would not be reduced.

Case 2. The Γ-reduction cancels γ. This case is the symmetric case of the left context. The rule m becomes $m: \tau_1\gamma_1\tau \rightarrow \sigma_1\sigma$. If γ_1 is not a piece, then $T\tau$ is reducible as proved by the equality $\gamma_1\tau\sigma^{-1}\sigma_1^{-1}\tau_1=\gamma_1 T^{-1}\delta^{-1}ABC$. Thus, γ_1 is a piece between rules m and l, and the critical pair is confluent:
On one hand $\beta\gamma T\tau \overset{*}{\underset{F}{\Rightarrow}} \beta\gamma_1\tau$. On the other hand $\alpha\delta T\tau \overset{*}{\underset{F\Gamma}{\Rightarrow}} \alpha ABC\gamma_1\tau$, because $\delta T \overset{*}{\underset{F\Gamma}{\Rightarrow}} ABC\gamma_1$ from **C**'(4) and Prop.2 applied to rule l. We conclude with $\alpha\delta T\tau \overset{*}{\underset{F\Gamma}{\Rightarrow}} \beta\gamma_1\tau$, by application of rule k ∎
When $T=1$, the word $\beta\gamma\tau$ is reduced by the rule $m: \tau_1\tau \rightarrow \sigma_1\sigma$, hence τ_1 and γ share a common suffix.

Lemma 8

Given a reduction of $\beta\gamma$ by right context without F-reduction, then

- *If the reduction is done by a piece between γ and the context, then we get a word ψ, F,Γ-irreducible, Γ-reducible only under the configuration* **LR** *(where $T=1$).*
- *Otherwise, the critical pair is confluent in this context.*

Proof.
Case 1. This suffix is a piece between l and m. Then $\tau_1\sigma_1^{-1}$ is a proper γ suffix. Otherwise the left member of l would be the concatenation of two pieces. This is impossible by assumption **C**'(4). We get a configuration, identical to **LR** except in the word T, presently empty. The conclusions still hold.
Case 2.The suffix is not a piece. Then τ_1 is a γ suffix, as in the opposite case, the identity $\delta^{-1}ABC\gamma=\tau\sigma^{-1}\sigma_1\sigma^{-1}\tau_1$ would imply the reducibility of rule k with the common suffix of β and C. We are allowed to write $\gamma=\gamma_1\tau_1$.

1) If γ_1 is the empty word, then the critical pair is confluent (same proof as in case 1 of lemma 6).
2) In the other case, γ is of the form $\gamma_2\sigma_1^{-1}\tau_1$ and $\beta\gamma\tau \overset{*}{\underset{F,\Gamma}{\Rightarrow}} \beta\gamma_2\sigma$. The identity deduced from the relations associated to rules l and m gives $\delta^{-1}ABC\gamma_2=\tau\sigma^{-1}$. This implies, by F-irreducibility of $\beta\gamma_2\sigma$ that either γ_2 or σ is empty.
   —If δ is empty then $\alpha\delta\tau=\alpha\delta\delta^{-1}ABC\gamma_2 \overset{*}{\underset{F}{\Rightarrow}} \alpha ABC\gamma_2 \overset{*}{\underset{F,\Gamma}{\Rightarrow}} \beta\gamma_2$. The critical pair is confluent.
   —If γ_2 is empty, then either τ or σ has a sufficiently large subword to initiate the confluence of the critical pair (details left out). This completes the proof of last lemma on reduction by contexts ∎

To resume the behaviour of a critical pair concatened to a context, we have two possibilities

—Either the confluence of the two members is possible.

—Or we get a word φ (resp. ψ) F-irreducible, Γ-reducible only with a configuration **LL** (resp. **LR**). The syntactical form of divergence points in reductions of any word is known. Of course, we have the configurations **RR** and **RL** corresponding to reductions of the other member of the critical pair $\alpha\delta$.

$$
\mathbf{RL}\left\{\begin{array}{ll}
k: M_1^{-1}\mu'_1\nu'^{-1}_1\alpha_2AB & \rightarrow \beta C^{-1}\\
l: BC\gamma & \rightarrow A^{-1}\rho\delta\\
m: \mu_1\mu'_1 & \rightarrow \nu_1\lambda\nu'_1\\
n: \lambda\alpha_2\rho & \rightarrow \omega\\
\text{Pieces: } \alpha_2, ABC, \lambda, & \mu'_1\nu'^{-1}_1, \rho.
\end{array}\right.
\qquad
\mathbf{RR}\left\{\begin{array}{ll}
k: \alpha\lambda AB & \rightarrow \beta C^{-1}\\
l: BC\gamma & \rightarrow A^{-1}\delta_2\sigma'^{-1}_1\tau'_1T_1^{-1}\\
m: \tau'_1\tau_1 & \rightarrow \sigma'_1\rho\sigma_1\\
n: \lambda\delta_2\rho & \rightarrow \omega\\
\text{Pieces: } \delta_2, & ABC, \sigma'^{-1}_1\tau'_1, \lambda, \rho.
\end{array}\right.
$$

4. Confluence of Dehn Algorithm

We first reformulate the fundamental result of small cancellation theory by means of symmetrized sets of rules. Let $\mathbf{G}=(G,S)$ be a finite group presentation where S is a ssr satisfying the conditions $\mathbf{C}'(6)$, or $\mathbf{C}'(4)$ and $\mathbf{T}(4)$. Then every word in $(G\cup G^{-1})^*$ equal in $\mathbf{G}$ to the unit element has 1 as unique F,Γ-irreducible form, where Γ is a symmetrized set of rules. The aim of this section is to prove this confluence property of words equal to the unit element.

In fact, we shall prove a more general result, based on the non-existence of configurations including **LL**, **LR**, **RL** and **RR** (cf. §3). Given an ordering, the symmetrization procedure computes a finite set of rules Γ described in section 2. We always suppose that the ssr S satisfies the condition $\mathbf{C}'(4)$. Let W be a word such that :

i) $W \overset{*}{\underset{F,\Gamma}{\Rightarrow}} 1$,

ii) $\exists W'\neq 1$, F,Γ*–irreducible*, such that $W \overset{*}{\underset{F,\Gamma}{\Rightarrow}} W'$.

Let us define two sets of descendants of W:

$$\Delta(W) = \{ Z \mid W \overset{*}{\underset{F,\Gamma}{\Rightarrow}} Z \} \text{ and } \Omega_0(W) = \{ Z \mid W \overset{*}{\underset{F,\Gamma}{\Rightarrow}} Z,\ \mathit{Irred}(Z)=\{1\} \}$$

where *Irred(Z)* is the non-empty set of Z's normal forms under the relation $\underset{F,\Gamma}{\Rightarrow}$. We localize a point of ***strong*** divergence in the dag of W's descendants.

Lemma 9

There exist two rules k and l from Γ and two words U, V with:

$$k : \alpha AB \longrightarrow \beta C^{-1} \qquad l : BC\gamma \longrightarrow A^{-1}\delta$$

where ABC is a piece between k and l, both U and V are F,Γ-irreducible, $Y = U\alpha ABC\gamma V$ belongs to $\Delta(W)$, $Y \xrightarrow[F,\Gamma]{*} Y_1 = U\beta\gamma V$ (or $U\alpha\delta V$) with $Irred(Y_1) = \{1\}$ and $Y \xrightarrow[F,\Gamma]{*} Y_2 = U\alpha\delta V$ (or $U\beta\gamma V$) with $1 \notin Irred(Y_2)$.

Proof. The following inclusions $\{1\} \subset \Omega_0(W) \subset \Delta(W)$ are strict from the hypothesis on W. As every element in $\Delta(W)$ is less or equal than W for our well-founded ordering, we may pick up a maximal element Z_0 in Ω_0. Soon, we have two words Z_1 and Z_2 with the following figure:

$$\begin{tikzcd}[column sep=small] & Z_0 \arrow[dl, "F{,}\Gamma"'] \arrow[dr, "F{,}\Gamma"] & \\ Z_1 \arrow[d, "*"] & & Z_2 \arrow[d, "*"] \\ 1 & & X \neq 1, irred \end{tikzcd}$$

Fig. 3

If $1 \in Irred(Z_2)$, then let $\Omega_1(W) = \Omega_0(W) \cap \Delta(Z_2)$. We have the following inclusions:

$$\{1\} \subset \Omega_1(W) \subset \Delta(W) \text{ and } \Omega_1(W) \subset \Omega_0(W)$$

as Z_2 satisfies the hypothesis upon W. By iterating the process, we built a decreasing chain of sets:

$$\Omega_0(W) \supset \Omega_1(W) \supset \cdots \supset \Omega_n(W) \supset \cdots$$

By noetherianity of reductions and finiteness of Γ, the set $\Omega_0(W)$ is finite and that the sequence of subsets is stationary from some index such that $\{1\} \not\subset Irred(Z_2)$. Thus, the following set is non-empty:

$$\Theta(W) = \{Z \mid Z \in \Delta(W),\ \exists Z_1, Z_2 \text{ s.t. } Z \xrightarrow[F,\Gamma]{} Z_i,\ \{1\} = Irred(Z_1) \text{ and } 1 \notin Irred(Z_2)\}.$$

Choose a minimal word Y in this set $\Theta(W)$ for the given ordering. The rules that reduce Y on Y_i, $i = 1,2$ do not belong to F: all the critical pairs of this set are resolved. These reductions are performed by a non-confluent critical pair of Γ. Otherwise, we would have the following figure:

$$\begin{tikzcd}[column sep=small] & Y = E\lambda_1 F\lambda_2 G \arrow[dl] \arrow[dr] & \\ E\rho_1 F\lambda_2 G \arrow[dr] & & E\lambda_1 F\rho_2 G \arrow[dl] \\ & E\rho_1 F\rho_2 & \end{tikzcd}$$

Fig. 4

By Prop.2, a non-confluent critical pair comes from a superposition on a piece between two relations. Thus, with the notations of I, we have $Y = U\alpha ABC\gamma V$ with two rules k and l. If the word U is reducible by a rule i from $F \cup \Gamma$, $U \xrightarrow[i]{} U'$, the following diagram contradicts the minimality of Y and concludes the proof ∎

$$\begin{tikzcd}[column sep=small]
 & & Y \arrow[dll, "k"'] \arrow[d, "i"] \arrow[drr, "l"] & & \\
U\beta C^{-1}C\gamma V \arrow[d, "i"] & & U'\alpha ABC\gamma V \arrow[dll, "k"'] \arrow[drr, "l"] & & U\alpha AA^{-1}\delta V \arrow[d, "i"] \\
U'\beta C^{-1}\gamma V & & & & U'\alpha AA^{-1}\delta V
\end{tikzcd}$$

Fig. 5

In this figure, the two words $\beta\gamma$ and $\alpha\delta$ behave symmetrically. We restrict ourselves to the case where $\beta\gamma$'s has 1 as unique normal form. After a step of Γ-reduction and perhaps some steps of free cancellation, the word Y reduces on $Y_1=U\beta\gamma V$ and on $Y_2=U\alpha\delta V$. As Y_1 can only reduce on the unit element and U, V, β, γ are F,Γ-irreducible, every reduction must start at either $U\beta$ or γV. Moreover, we can detail the structure of U and V with respect to reductions, as these two words are entirely cancelled:

Lemma 10
The structure of the words U and V is expressed by the identities:

$$U = \mu_n M_n \cdots \mu_1 M_1$$
$$V = T_1\tau_1 \cdots T_m\tau_m$$

where the μ_i (resp. τ_j) are prefix (resp. suffix) of left members of rules m_i (resp. n_j) in Γ, non-empty except possibly μ_n (resp. τ_m), such that there exists a reduction path from Y_1 to 1 whose subsequence of Γ-reductions contains the subsequence $(m_i)_{i=1\cdots n}$ (resp. $(n_j)_{j=1\cdots m}$) whose each element reduces the corresponding word μ_i (resp. τ_j) in U (resp. V), the words M_i (resp. T_j) expressing possible free cancellations.

We now precisely analyze this sequence of reductions. The first reduction steps operate on $\mu_1 M_1\beta\gamma$ or on $\beta\gamma T_1\tau_1$. None of these two possible reductions gives the confluence of the critical pair as the two reduction dags are totally disconnected. From lemmas 4 to 8, the first reductions give the words φ and ψ. Our second assumption (after the C'(4) condition) asserts the non-existence in Γ of the configurations of the previous section:

Hypothesis 2 : none of the configurations **LL**, **LR**, **RL** and **RR** appears in Γ.

Observe that by hypothesis 1 no **L** neither **R** configuration exist. The words φ and ψ being irreducible, the next rewriting step reduces either a U suffix or a V prefix. When these reductions occur on the side of $\beta\gamma$ opposed to the first reduction side, we get a word $\xi=\nu_1\beta_2\gamma_2\sigma_1$ F-irreducible. Thus, we examine the remaining cases: $\mu_2 M_2\nu_1\beta_2\gamma$ and $\beta\gamma_2\sigma_1 T_2\tau_2$.

The Case $\mu_2 M_2\nu_1\beta_2\gamma$. First, M_2 doesn't freely cancel all ν_1. Otherwise, the rule m_1: $\mu_1\mu_1' \rightarrow \nu_1\nu_1'$ where $\mu_1'\nu'^{-1}_1$ is a piece shows that $M_2\mu_1$, with the subword $\nu_1^{-1}\mu_1$ reducible on $\nu_1'\mu_1'^{-1}$ would be reducible. But U is F,Γ-irreducible. Next, with m_2: $\mu_2\mu_2' \rightarrow \nu_2\nu_2'$, we have the identity:

$$\nu_1=M_2^{-1}\mu_2'\nu_2'^{-1}\beta_3 \text{ where } \beta_3\neq 1 \text{ and } \mu_2'\nu_2'^{-1} \text{ is a piece.}$$

Suppose a contrario that there exists a prefix P of $\mu_2'\nu_2'^{-1}$ such that $\nu_1=M_2^{-1}P$. Two relations associated to m_1 and m_2 are $P\nu'_1\mu'^{-1}_1\mu_1^{-1}M_2^{-1}$ and $\mu'_2\nu'^{-1}_2\nu_2^{-1}\mu_2$. Then P is not a piece, otherwise the context U would be reducible by $M_2\mu_1 \xrightarrow[F,\Gamma]{*} P\nu'_1\mu'^{-1}_1$. Thus, the two relations are syntactically equal, this implies first $M_2=1$, second $\mu_2\mu_1$, subword of U, would be F-reducible. Clearly, all these cases are impossible, the existence of the non-empty word β_3 and of the piece $\mu'_2\nu'^{-1}_2$ follows. And we have $\mu_2 M_2 \nu_1 \beta_2 \gamma \overset{*}{\underset{F,\Gamma}{\Rightarrow}} \nu_2\beta_3\beta_2\gamma=\zeta$.

The Case $\beta\gamma_2\sigma_1 T_2\tau_2$. With the previous reasoning, T_2 does not freely cancel all the word σ_1. And if $n_2 : \tau'_2\tau_2 \rightarrow \sigma'_2\sigma_2$ then $\sigma_1=\gamma_3\sigma'^{-1}_2\tau'_2 T_2^{-1}$. Note that the relations defined by the two rules n_i, $i=1,2$, are presently:

$$T_2^{-1}\tau_1^{-1}\tau'^{-1}_1\sigma'_1 P \text{ (where } P \text{ is a suffix of } \sigma'^{-1}_2\tau'_2) \quad \text{and} \quad \tau_2\sigma_2^{-1}\sigma'^{-1}_2\tau'_2.$$

And the subword $\beta\gamma_2$ remains unmodified by this new reduction:

$$\beta\gamma_2\sigma_1 T_2\tau_2 \overset{*}{\underset{F,\Gamma}{\Rightarrow}} \beta\gamma_2\gamma_3\sigma_2=\chi.$$

Thus in any case, the second step of Γ-reduction does not affect the the β and γ's of the previous Γ-reduction. We now turn to the Γ-reducibility of the three words ζ,ξ and χ. The three cases will be successively detailed. We first remember the configuration that yields the word, and summarize the various possible reductions in a table, whose leftmost column contains the reducing configurations. The last rule in each configuration reduces a subword of the three words ζ,ξ and χ, subword intersecting at least two successive components. If one of its left members components is not a piece, the identical relation is displayed in the second column. The last one reports a consequence of this identity: a rule includes a freely reducible subword or possesses a non-empty prefix or suffix common to both its left and right members. Thus all components of the last rule left member are pieces.

Reduction of ξ.

$$\left\{\begin{array}{rcl} \alpha AB & \rightarrow & M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2 C^{-1} \\ BC\gamma_2\sigma'^{-1}_1\tau'_1 T_1^{-1} & \rightarrow & A^{-1}\delta \\ \mu_1\mu'_1 & \rightarrow & \nu_1\nu'_1 \\ \tau'_1\tau_1 & \rightarrow & \sigma'_1\sigma_1 \end{array}\right.$$

Configuration	Identifications	Consequences
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\rho\beta_2 C^{-1}$ $\mu_1\mu'_1 \to \nu_1\lambda\nu'_1$ $\lambda\rho \to \omega$	$\lambda\rho\ldots=\lambda\nu'_1\mu'^{-1}_1\ldots$ $\rho\ldots\lambda=\rho\ldots\mu'_1\nu'^{-1}_1$	rule 1 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2\lambda C^{-1}$ $BC\rho\gamma_2\sigma'^{-1}_1\tau'_1 T_1^{-1} \to A^{-1}\delta$ $\lambda\rho \to \omega$	$\lambda C^{-1}B^{-1}\ldots=\lambda\rho\ldots$ $\rho\ldots\lambda=\rho\ldots BC$	rule 2 n.r. rule 1 n.r.
$BC\gamma_2\lambda\sigma'^{-1}_1\tau'_1 T_1^{-1} \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\rho\sigma$ $\lambda\rho \to \omega$	$\lambda\rho\ldots=\lambda\sigma'^{-1}_1\tau'_1\ldots$ $\rho\ldots\lambda=\rho\ldots\tau'^{-1}_1\sigma'_1$	rule 2 n.r. rule 1 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2 C^{-1}$ $BC\rho\gamma_2\sigma'^{-1}_1\tau'_1 T_1^{-1} \to A^{-1}\delta$ $\mu_1\mu'_1 \to \nu_1\lambda\nu'_1$ $\lambda\beta_2\rho \to \omega$	$\lambda\beta_2\ldots=\lambda\nu'_1\mu'^{-1}_1\ldots$ $\beta_2\rho\ldots=\beta_2 C^{-1}B^{-1}\ldots$ $\rho\ldots\beta_2=\rho\ldots BC$	rule 3 n.r. rule 2 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2\lambda C^{-1}$ $BC\gamma_2\sigma'^{-1}_1\tau'_1 T_1^{-1} \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\rho\sigma$ $\lambda\gamma_2\rho \to \omega$	$\lambda\gamma_2\ldots=\lambda C^{-1}B^{-1}\ldots$ $\gamma_2\rho\ldots=\gamma_2\sigma'^{-1}_1\tau'_1\ldots$ $\rho\ldots\gamma_2=\rho\ldots\tau'^{-1}_1\sigma'_1$	rule 1 n.r. rule 3 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2 C^{-1}$ $BC\gamma_2\sigma'^{-1}_1\tau'_1 T_1^{-1} \to A^{-1}\delta$ $\mu_1\mu'_1 \to \nu_1\lambda\nu'_1$ $\tau'_1\tau_1 \to \sigma'_1\rho\sigma$ $\lambda\beta_2\gamma_2\rho \to \omega$	$\lambda\beta_2\ldots=\lambda\nu'_1\mu'^{-1}_1\ldots$ $\beta_2\gamma_2\ldots=\beta_2 C^{-1}B^{-1}\ldots$ $\gamma_2\rho\ldots=\gamma_2\sigma'^{-1}_1\tau'_1\ldots$ $\rho\ldots\gamma_2=\rho\ldots\tau'^{-1}_1\sigma'_1$	rule 1 n.r. rule 2 n.r. rule 4 n.r. rule 2 n.r.

All configurations but the last one are **L**, **R**, **LL**, **LR** configurations.

Reduction of $\boldsymbol{\gamma\zeta}$.

$$\left\{\begin{array}{ccl} \alpha AB & \to & M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2 C^{-1} \\ BC\gamma & \to & A^{-1}\delta \\ \mu_1\mu'_1 & \to & M_2^{-1}\mu'_2\nu'^{-1}_2\beta_3\nu'_1 \\ \mu_2\mu'_2 & \to & \nu_2\nu'_2 \end{array}\right.$$

Configuration	Identifications	Consequences
$\mu_1\mu'_1 \to M_2^{-1}\mu'_2\nu'^{-1}_2\rho\beta_3\nu'_1$ $\mu_2\mu'_2 \to \nu_2\lambda\nu'_2$ $\lambda\rho \to \omega$	$\lambda\rho\ldots=\lambda\nu'_2\mu'^{-1}_2\ldots$ $\rho\ldots\lambda=\rho\ldots\mu'_2\nu'^{-1}_2$	rule 1 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\rho\beta_2C^{-1}$ $\mu_1\mu'_1 \to M_2^{-1}\mu'_2\nu'^{-1}_2\beta_3\lambda\nu'_1$ $\lambda\rho \to \omega$	$\lambda\rho\ldots=\lambda\nu'_1\mu'^{-1}_1\ldots$ $\rho\ldots\lambda=\rho\ldots\mu'_1\nu'^{-1}_1$	rule 1 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2\lambda C^{-1}$ $BC\rho\gamma \to A^{-1}\delta$ $\lambda\rho \to \omega$	$\lambda\rho\ldots=\lambda C^{-1}B^{-1}\ldots$ $\rho\ldots\lambda=\rho\ldots BC$	rule 2 n.r. rule 1 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\rho\beta_2C^{-1}$ $\mu_1\mu'_1 \to M_2^{-1}\mu'_2\nu'^{-1}_2\beta_3\nu'_1$ $\mu_2\mu'_2 \to \nu_2\lambda\nu'_2$ $\lambda\beta_3\rho \to \omega$	$\lambda\beta_3\ldots=\lambda\nu'_2\mu'^{-1}_2\ldots$ $\beta_3\ldots\lambda=\beta_3\ldots\mu'_2\nu'^{-1}_2$ $\rho\ldots\beta_3=\rho\ldots\mu'_1\nu'^{-1}_1$	rule 2 n.r. rule 3 n.r. rule 2 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2C^{-1}$ $BC\rho\gamma \to A^{-1}\delta$ $\mu_1\mu'_1 \to M_2^{-1}\mu'_2\nu'^{-1}_2\beta_3\lambda\nu'_1$ $\lambda\beta_2\rho \to \omega$	$\lambda\beta_2\ldots=\lambda\nu'_1\mu'^{-1}_1\ldots$ $\beta_2\ldots\lambda=\beta_2\ldots\mu'_1\nu'^{-1}_1$ $\rho\ldots\beta_2=\rho\ldots BC$	rule 1 n.r. rule 3 n.r. rule 1 n.r.
$\alpha AB \to M_1^{-1}\mu'_1\nu'^{-1}_1\beta_2C^{-1}$ $BC\rho\gamma \to A^{-1}\delta$ $\mu_1\mu'_1 \to M_2^{-1}\mu'_2\nu'^{-1}_2\beta_3\nu'_1$ $\mu_2\mu'_2 \to \nu_2\lambda\nu'_2$ $\lambda\beta_3\beta_2\rho \to \omega$	$\lambda\beta_3\ldots=\lambda\nu'_2\mu'^{-1}_2\ldots$ $\beta_3\beta_2\ldots=\beta_3\nu'_1\mu'^{-1}_1\ldots$ $\beta_2\rho\ldots=\beta_2C^{-1}B^{-1}\ldots$ $\rho\ldots\beta_2=\rho\ldots BC$	rule 3 n.r. rule 1 n.r. rule 2 n.r. rule 1 n.r.

The last configuration and the first four rules one are new.
Reduction of χ.

$$\left\{\begin{array}{rcl} \alpha AB & \to & \beta C^{-1} \\ BC\gamma_2\sigma'^{-1}_1\tau'_1T_1^{-1} & \to & A^{-1}\delta \\ \tau'_1\tau_1 & \to & \sigma'_1\gamma_3\sigma'^{-1}_2\tau'_2T_2^{-1} \\ \tau'_2\tau_2 & \to & \sigma'_2\sigma_2 \end{array}\right.$$

Configuration	Identifications	Consequences
$\alpha AB \to \beta\lambda C^{-1}$ $BC\rho\gamma_2\sigma'^{-1}_1\tau'_1T^{-1}_1 \to A^{-1}\delta$ $\lambda\rho \to \omega$	$\lambda\rho ... = \lambda C^{-1}B^{-1}...$ $\rho ... \lambda = \rho ... BC$	rule 2 n.r. rule 1 n.r.
$BC\gamma_2\lambda\sigma'^{-1}_1\tau'_1T^{-1}_1 \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\rho\gamma_3\sigma'^{-1}_2\tau'_2T^{-1}_2$ $\lambda\rho \to \omega$	$\lambda\rho ... = \lambda\sigma'^{-1}_1\tau'_1 ...$ $\rho ... \lambda = \rho ... \tau'^{-1}_1\sigma'_1$	rule 2 n.r. rule 1 n.r.
$\tau'_1\tau_1 \to \sigma'_1\gamma_3\lambda\sigma'^{-1}_2\tau'_2T^{-1}_2$ $\tau'_2\tau_2 \to \sigma'_2\rho\sigma_2$ $\lambda\rho \to \omega$	$\lambda\rho ... = \lambda\sigma'^{-1}_2\tau'_2 ...$ $\rho ... \lambda = \rho ... \tau'^{-1}_2\sigma'_2$	rule 2 n.r. rule 1 n.r.
$\alpha AB \to \beta\lambda C^{-1}$ $BC\gamma_2\sigma'^{-1}_1\tau'_1T^{-1}_1 \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\rho\gamma_3\sigma'^{-1}_2\tau'_2T^{-1}_2$ $\lambda\gamma_2\rho \to \omega$	$\lambda\gamma_2 ... = \lambda C^{-1}B^{-1}...$ $\gamma_2\rho ... = \gamma_2\sigma'^{-1}_1\tau'_1 ...$ $\rho ... \gamma_2 = \rho ... \tau'^{-1}_1\sigma'_1$	rule 2 n.r. rule 3 n.r. rule 2 n.r.
$BC\gamma_2\lambda\sigma'^{-1}_1\tau'_1T^{-1}_1 \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\gamma_3\sigma'^{-1}_2\tau'_2T^{-1}_2$ $\tau'_2\tau_2 \to \sigma'_2\rho\sigma_2$ $\lambda\gamma_3\rho \to \omega$	$\lambda\gamma_3 ... = \lambda\sigma'^{-1}_1\tau'_1 ...$ $\gamma_3\rho ... = \gamma_3\sigma'^{-1}_2\tau'_2 ...$ $\rho ... \gamma_3 = \rho ... \tau'^{-1}_2\sigma'_2$	rule 2 n.r. rule 3 n.r. rule 2 n.r.
$\alpha AB \to \beta\lambda C^{-1}$ $BC\gamma_2\sigma'^{-1}_1\tau'_1T^{-1}_1 \to A^{-1}\delta$ $\tau'_1\tau_1 \to \sigma'_1\gamma_3\sigma'^{-1}_2\tau'_2T^{-1}_2$ $\tau'_2\tau_2 \to \sigma'_2\rho\sigma_2$ $\lambda\gamma_2\gamma_3\rho \to \omega$	$\lambda\gamma_2 ... = \lambda C^{-1}B^{-1}...$ $\gamma_2\gamma_3 ... = \gamma_2\sigma'^{-1}_1\tau'_1 ...$ $\gamma_3\rho ... = \gamma_3\sigma'^{-1}_2\tau'_2 ...$ $\rho ... \gamma_3 = \rho ... \tau'^{-1}_2\sigma'_2$	rule 2 n.r. rule 3 n.r. rule 4 n.r. rule 3 n.r.

We interpret configurations geometrically by S-diagrams. All configurations with the same number of rules give the same diagram. Paths in diagrams are orientated. Moreover, they are ordered as words, with the ordering used in the symmetrization. A graph associated with a configuration of n+3 rules n>0 will be denoted by C_n. The boudary cycle of each region is a relation from the ssr S. It splits in two non-empty disjoint paths, an interior path and an exterior one. This latter path belongs to a single region and will be labeled clockwise with respect to this region, the interior path being labeled counter-clockwise. With this convention, the associated words are equal in **G** and the hypothesis imply that they are comparable by the ordering. The smaller one is represented by a dashed line. Let us recall that an interior edge, thus belonging to two distinct cycles, represents a piece between two relations. The first diagram C_0 encodes all **L** and **R** configurations:

$$\mathbf{L}\begin{cases} \alpha AB \to \beta\lambda C^{-1} \\ BC\rho\gamma \to A^{-1}\delta \\ \lambda\rho \to \omega \end{cases} \quad \mathbf{R}\begin{cases} \alpha\lambda AB \to \beta C^{-1} \\ BC\gamma \to A^{-1}\rho\delta \\ \lambda\rho \to \omega \end{cases}$$

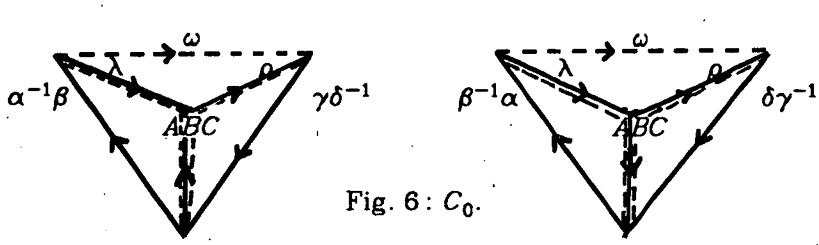

Fig. 6 : C_0.

We give the remaining configurations without labeling, and with the convention that the upper cycle is the last rule, ω being its exterior edge, its neighbours being λ and ρ.

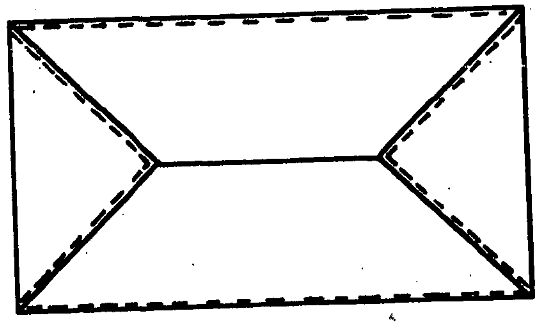

Fig. 7: C_1.

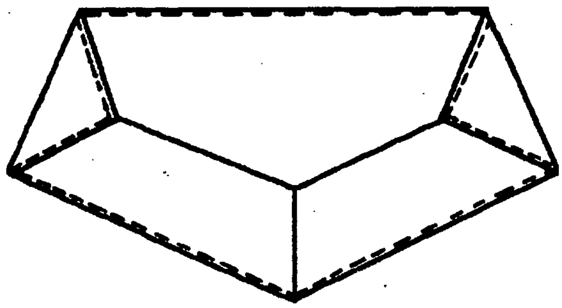

Fig. 8: C_2.

The following diagrams are composed of n four-gons such that the three interior edges path is greater than the exterior one, of two triangles whose exterior edge is greater than the concatenation of its two interior ones, and finally of a closing cycle with $n+2$ consecutive interior edges, whose concatenation is greater than the exterior path ω. We give an example of labeled C_4 corresponding to the reduction of $\beta_3\beta_2\gamma_2\gamma_3$:

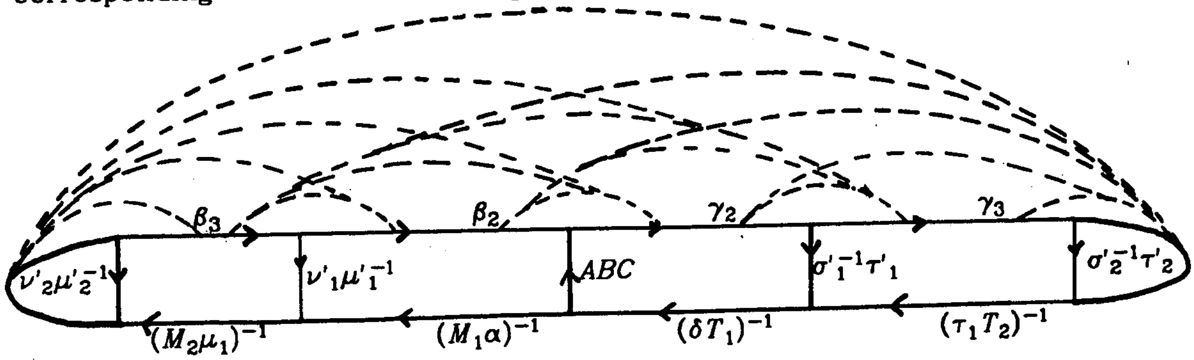

Fig. 9: O_4.

All forbidden C_i, $i=0,\ldots,4$, are represented on this graph. Thus, our third assumption, which includes as a special case the Hypothesis 2, is:

Hypothesis 3 : No diagram C_n, $n\geq 0$, exist for the given ordering.

Thus the three words ζ, ξ, χ are F,Γ-irreducible. Under this assumption, this creation of irreducible subwords goes on:

Lemma 11
The word Y_1 *reduces on* $Y_3=\nu_n\beta_{n-1}\cdots\beta_2\gamma_2\cdots\gamma_{m-1}\sigma_m$, *with for* $i=2\cdots n-1$ *and* $j=2\cdots m-1$:

i) $\beta_i\neq 1$ *and* $\gamma_j\neq 1$.

ii) *The word* Y_3 *is* F,Γ*-irreducible.*

Proof. By induction on $n+m$. The induction hypothesis is the structure of the previous rules and the irreducibility of the word still reduced:

$$\begin{cases} n_l : \tau'_l\tau_l \longrightarrow \sigma'_l\gamma_l\sigma'^{-1}_{l+1}\tau'_{l+1}T^{-1}_{l+1} \\ m_k : \mu'_k\mu_k \longrightarrow M^{-1}_{k+1}\mu'_{k+1}\nu'^{-1}_{k+1}\beta_k\nu'_k \end{cases}$$

where $\beta_k\neq 1$ and $\gamma_l\neq 1$, $\mu'_{k+1}\nu'^{-1}_{k+1}$ and $\sigma'^{-1}_{l+1}\tau'_{k+1}$ are pieces. The word $Y_{k,l}=\nu_k\beta_{k-1}\cdots\beta_2\gamma_2\cdots\gamma_{l-1}\sigma_l$ being irreducible. We have seen that this hypothesis holds for rules k,l,m_1 and n_1. And the next reductions must freely cancel either the word M_{k+1} or T_{l+1}. After what the reduction by either rule m_{k+1} or n_{l+1} may occur. The description of cases $\mu_2M_2\nu_1\beta_2\gamma$ and $\beta\gamma_2\sigma_1T_2\tau_2$ shows that these two new rules have the previous form. Then, the word $Y_{k+1,l+1}=\nu_{k+1}\beta_k\cdots\beta_2\gamma_2\cdots\gamma_l\sigma_{l+1}$ is irreducible by the non-existence of the configurations C_i, $i=1\cdots\max(k+1,l+1)$. Note that the deductions using the non-confluence of Y_j, j=1,2, are now replaced by the irreducibility of both $M_{k+1}\mu_k$ and $\tau_l T_{l+1}$ ∎

We have proved that the word Y_1 reduces on a non-null word Y_3, F,Γ-irreducible, in contradiction with the construction of Y. Therefore, under hypothesis 1 (C'(4)) and 3 ($C_j, j=1\cdots$), if a word W reduces on 1, then 1 is its only irreducible form:

$$\forall W,\ 1\in Irred(W) \Longrightarrow Irred(W)=\{1\}$$

Let W be a word such that $W=_G 1$. There exist words W' and $T_1,\ldots,T_k$ such that $W'=T_1R_1T_1^{-1}\cdots T_kR_kT_k^{-1}$, where $R_i\in S$, the ssr of defining relations. Moreover W is computed from W' by adjunction or deletion of F-redexes. We thus have the following diagram:

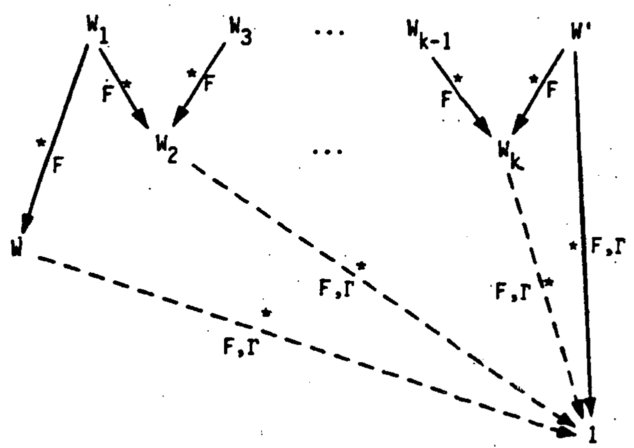

Fig. 10

which concludes the proof $\forall W,\ W=_G 1 \Longrightarrow Irred(W)=\{1\}$.

Theorem 12

Let $\mathbf{G}$*=(G,S) be a finite group presentation satisfying* $\mathbf{C}'(4)$ *and without S-diagram* $C_i, \forall i \in \mathbf{N}$*, for some ordering, then the symmetrized presentation of* $\mathbf{G}$ *computed with this ordering solves the word problem for* $\mathbf{G}$.

Corollary 13

If the presentation $\mathbf{G}$ *satisfies either* $\mathbf{C}'(6)$ *or* $\mathbf{C}'(4)$ *and* $\mathbf{T}(4)$*, then Dehn algorithm solves its word problem.*

Proof. As $\mathbf{G}$ satisfies $\mathbf{C}'(4)$, if $\mathbf{G}$ also satisfies $\mathbf{T}(4)$, then no C_i diagram exists as they all include an interior vertex surrouded by three consecutive regions. Observe that $\mathbf{T}(4)$ was not sufficient in the proofs of lemmas 5-8. If $\mathbf{G}$ satisfies $\mathbf{C}'(6)$, then by $\mathbf{C}'(4)$, C_0 is prohibited, while all the other ones include three consecutive pieces reducing to the rest of the elementary circuit or relation (i.e to the words $M_{i+1}\mu_i$, $M_1\alpha$, δT_1 and $\tau_j T_{j+1}$). This is impossible by $\mathbf{C}'(6)$ as the length of three consecutive pieces in a relation is strictly smaller than the remaining part of this relation ▪

5. Examples

The present proof was initiated by H.Bücken [Büc79b]. He began an analysis of the symmetrization, isolated the words Y, and gave the irreducible form of Y_1. The present proof profits from experimentations done with an implementation of the symmetrization [LeC84]. A thorough analysis of symmetrization was possible, implying the discovery of diagrams. The contexts, while indexed by $\mathbb{N}$, are finitely generated by the symmetrized presentation, also their non-existence can be checked.

Now, we may discuss the hypothesis of the main theorem. There are two of them, first $\mathbf{C}'(4)$, second the non-existence of C_i configurations. As the first assumption implies the non-existence of C_0, we may ask whether $\mathbf{C}'(4)$ is only used for this purpose? The answer is no, as shown by the following symmetrized set of the group $G_1=(a,b;ababa^{-1}b^{-1})$:

$$\left\{\begin{array}{rclrcl} a^{-1}ba & \rightarrow & bab & b^{-1}a^{-1}b & \rightarrow & aba^{-1} \\ bab^{-1} & \rightarrow & aba & abab & \rightarrow & ba \\ a^{-1}b^{-1}a^{-1} & \rightarrow & ba^{-1}b^{-1} & aba^{-1}b^{-1} & \rightarrow & b^{-1}a^{-1} \\ b^{-1}a^{-1}b^{-1} & \rightarrow & a^{-1}b^{-1}a & ba^{-1}b^{-1}a & \rightarrow & a^{-1}b^{-1} \\ b^{-1}ab & \rightarrow & ab^{-1}a^{-1} & baba^{-1} & \rightarrow & a^{-1}b \end{array}\right.$$

The presentation is $\mathbf{C}'(2)$, but not $\mathbf{C}'(3)$. Hence it does not satisfy the first hypothesis. However, it satisfies $\mathbf{T}(4)$ as every relation begins with a letter and ends with the other one. Thus no C_i configuration appears in its Cayley diagram. However, Dehn algorithm does not solve its word problem. Thus the condition $\mathbf{C}'(4)$ is necessary. If we take the same notations as in the proof, we have for example:

$$\begin{tikzcd}[column sep=small]
Z=aabbb^{-1}abba^{-1}b \arrow[d] \arrow[dr] & & \\
aababba^{-1}b^{-1} \arrow[ddd] & Y=aabbab^{-1}a^{-1}ba^{-1}b^{-1} \arrow[d] \arrow[dr] & \\
& Y_1=aababaa^{-1}ba^{-1}b^{-1} \arrow[d] & Y_2=aabbaaba^{-1}a^{-1}b^{-1} \\
& aababba^{-1}b^{-1} \arrow[dl] & \\
ababa^{-1}b^{-1} \arrow[d] & & \\
baa^{-1}b^{-1} \arrow[d] & & \\
1 & &
\end{tikzcd}$$

Fig. 11

The word Z both reduces to 1 and Y_2, irreducible. But Z is not minimal, Y is minimal (cf. beginning of prev. §) and the corresponding diagram is

$$\begin{cases} k : bab^{-1} \longrightarrow aba \\ l : b^{-1}a^{-1}b \longrightarrow aba^{-1} \\ m_1 : abab \longrightarrow ba \\ m_2 : abab \longrightarrow ba \end{cases}$$

The original piece is $b^{-1}a^{-1}$ with only $A=1$, then $\beta=ab=\mu_1=\mu'_1$. The point is that, no word $\nu_1\beta_2\gamma$ exists, as β is entirely canceled by the m_1 reduction. The rule m_1 has its left member composed of the juxtaposition of two pieces, contradicting $C'(4)$. The diagram has an interior vertex of degree four:

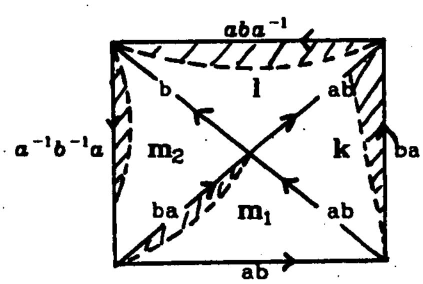

Fig. 12

The right members of the rules are delimited by the dashed areas. This example shows clearly our use of condition $C'(4)$. The case $\lambda\rho \rightarrow \omega$, where both λ and ρ are pieces occurs with the rule l. The non-existence of such rules is constantly use in the study of critical pair reduction by a context. The other application of $C'(4)$, including the previous one as special case, is:

C′(4): if $P_1P_2U \in S$, where P_1,P_2 are pieces, then $U^{-1} \underset{F,I}{\overset{*}{\Rightarrow}} P_1P_2$.

This proposition is an assumption on diagrams rather than a diagram itself.

Another example is the group $G_2=(A,B,C;ABC=CBA)$ whose one symmetrized set is:

$$\left\{\begin{array}{rcl rcl}
CBA & \rightarrow & ABC & C^{-1}AB & \rightarrow & BAC^{-1} \\
A^{-1}CB & \rightarrow & BCA^{-1} & ABCA^{-1} & \rightarrow & CB \\
B^{-1}A^{-1}C & \rightarrow & CA^{-1}B^{-1} & BCA^{-1}B^{-1} & \rightarrow & A^{-1}C \\
C^{-1}B^{-1}A^{-1} & \rightarrow & A^{-1}B^{-1}C^{-1} & CA^{-1}B^{-1}C^{-1} & \rightarrow & B^{-1}A^{-1} \\
B^{-1}C^{-1}A & \rightarrow & AC^{-1}B^{-1} & BAC^{-1}B^{-1} & \rightarrow & C^{-1}A
\end{array}\right.$$

This group is **C**′(5) but not **C**′(6). And it does not satisfy **T**(4). Then, we have for example the following non-confluent derivations [Büc79a]:

$$ABABABCA^{-1}CBCA^{-1}B^{-1}A^{-1}B^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$

$$\swarrow \qquad\qquad\qquad\qquad \downarrow$$

$$ABABABCBCA^{-1}CA^{-1}B^{-1}AB^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$

$$ABABABCA^{-1}CA^{-1}CA^{-1}B^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$ABABABCA^{-1}CA^{-1}B^{-1}A^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$ABABCBCA^{-1}B^{-1}A^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$ABABCA^{-1}CA^{-1}B^{-1}C^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$ABABCA^{-1}B^{-1}A^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$ABCA^{-1}B^{-1}C^{-1}$$
$$\downarrow$$
$$CBB^{-1}C^{-1}$$
$$\downarrow$$
$$1$$

Fig. 13

The associated diagram is a C_2 one, the last rule being a $\lambda\beta_2\gamma_2\rho \rightarrow \omega$.

$$\left\{\begin{array}{ll}
k : A^{-1}CB & \rightarrow BCA^{-1} \\
l : BCA^{-1}B^{-1} & \rightarrow A^{-1}C \\
m_1 : ABCA^{-1} & \rightarrow CB \\
n_1 : CA^{-1}B^{-1}C^{-1} & \rightarrow B^{-1}A^{-1} \\
o : BCA^{-1}B^{-1} & \rightarrow A^{-1}C
\end{array}\right.$$

The last rule being a $\lambda\alpha_2\delta_2\rho \rightarrow \omega$ one. The corresponding graph is therefore:

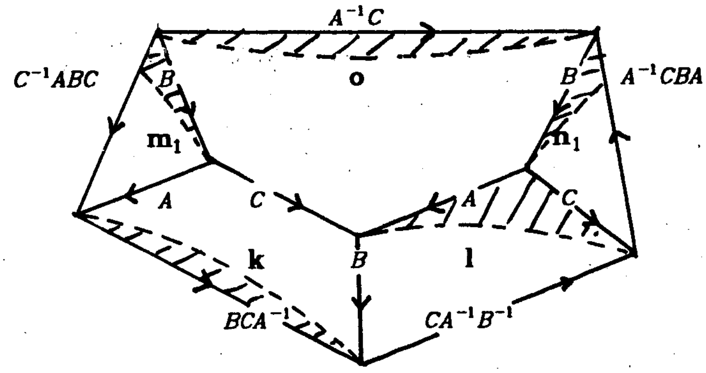

Fig. 14

However, the following symmetrized set for G_2 is also canonical: all its critical pairs are resolved. It is confluent on all words.

$$\left\{\begin{array}{lcc} ABC & \to & CBA \\ C^{-1}B^{-1}A^{-1} & \to & A^{-1}B^{-1}C^{-1} \\ BAC^{-1} & \to & C^{-1}AB \\ CA^{-1}B^{-1} & \to & B^{-1}A^{-1}C \\ A^{-1}CB & \to & BCA^{-1} \\ B^{-1}C^{-1}A & \to & AC^{-1}B^{-1} \end{array}\right.$$

Its noetherianity is proved by induction on word length: at most one reduction reduces the first generator, and the rules are not length-increasing. The fact that the present proof applies to this symmetrized set follows from the ordering which is non-length-increasing, from the fact that all superpositions occur on non-pieces, thus no C_i diagram exists, pieces have length 1 while the relation has length 6.

In constrat to the usual geometric proof [Lyn77,ch.5], the present one deals only with elementary properties of words. So does the proof by Greendlinger [Gre60a,b] based on the reduction F_G. In the reduction dag of a word, we have localized a point of divergence, so that no numerical result relative to the original word can be given such as Greendlinger lemma. However, localization exhibits critical diagrams.

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